GEOMETRIC ASPECTS OF THE ABELIAN MODULAR FUNCTIONS OF GENUS FOUR (I)

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1. *Introduction.*—The plane curve of genus 4 has a canonical series g_3^6 and is mapped from the plane by the canonical adjoints into the *normal* curve of genus 4, a space sextic which is the complete intersection of a quadric and a cubic surface. If we denote a point of this quadric by the parameters t, τ of the cross generators through it the equation of this sextic is $F = (a\tau)^3 (\alpha t)^3 = 0$. For geometric purposes we may define a modular function to be any rational or irrational invariant of the form F, bi-cubic in the digredient binary variables τ, t; for transcendental purposes it is desirable to restrict this definition by requiring further that this invariant, regarded as a function of the normalized periods ω_{ij} of the abelian integrals attached to the curve, be uniform.

There seems to be an unusually rich variety of geometric entities which center about this normal curve. Some of these have received independent investigation. It is the purpose of this series of abstracts to indicate a number of new relations among these various entities and to connect each with the normal sextic F. The methods employed are in the main geometric. Direct algebraic attack on problems which contain nine irremovable constants, or moduli, is difficult. However much information is gained by a free use of algebraic forms containing sets of variables drawn from different domains. Both finite and infinite discontinuous groups are utilized at various times.

2. *The Figure of Two Space Cubic Curves.*—White[2] has introduced for other purposes the interpretation of the form $F = 0$ as the incidence condition of the point τ of the space cubic curve $C_1(\tau)$ and the plane t of the space cubic $C_2(t)$. There is dually an incidence condition of plane τ of $C_1(\tau)$ and point t of $C_2(t)$, expressed by a form $\overline{F} = (A\tau)^3 (\mathrm{A}t)^3 = 0$. We call the sextics of genus 4 determined by $F = 0$ and $\overline{F} = 0$ *reciprocal.* Each is the same covariant of degree three of the other.

3. *A. Set of Four Mutually Related Rational Plane Sextics.*—On each of the cubic curves $C_1(\tau)$, $C_2(t)$ regarded as a point locus there is a net of point quadrics Q_1, Q_2, respectively; on each regarded as a plane locus there is a net of quadric envelopes, $\overline{Q}_1$, $\overline{Q}_2$, respectively. The net Q_1 cuts the curve $C_2(t)$ in ∞^2 sets of six points which lie in an $I_2^6(t)$. An I_2^6 on a binary domain may be visualized as the line sections of a projectively definite (but not localized) rational plane sextic $S_2(t)$. Thus the four nets determine the four rational plane sextics of the array

$$S_1(\tau),\ S_2(t)$$
$$\bar{S}_1(\tau),\ \bar{S}_2(t).$$

Two sextics in a row of the array will be called *paired sextics;* two in a column, *counter sextics;* and the other pairs, *diagonal sextics.* If any one of these sextics be given, its I_2^6 spread out on a space cubic will determine the other space cubic and thereby the entire set of four. The nodal parameters of the paired sextics in the upper row are those of the ten common chords of C_1, C_2; in the lower row, those of the ten common axes of C_1, C_2. The equations of the sextics are

$$(\alpha\alpha')^2(\alpha t)(\alpha' t)(a\tau)^3(a'\tau)^3 = 0,\quad (AA')^2(A\tau)(A'\tau)(\mathrm{A}t)^3(\mathrm{A}'t)^3 = 0,$$
$$(AA')^2(\mathrm{A}t)(\mathrm{A}'t)(A\tau)^3(A'\tau)^3 = 0,\quad (aa')^2(a\tau)(a'\tau)(\alpha t)^3(\alpha' t)^3 = 0.$$

Here the coefficients of the quadratics in t or τ furnish three line sections of the respective sextic. The significance of the quadratic parameter appears in 6.

4. *Two Birationally Related Quartic Surfaces.*—The two nets Q_1, Q_2 of point quadrics on C_1, C_2, respectively, are apolar to a web of quadric envelopes $\bar{Q}$; similarly the nets $\bar{Q}_1$, $\bar{Q}_2$ are apolar to a web of point quadrics, Q. The jacobians, $\bar{J}$, J, of these respective webs are quartic envelope or surface, respectively; the first on the ten common chords, the second on the ten common axes of C_1, C_2. If we map by means of the web Q its space upon another space, the jacobian J, the locus of nodes of quadrics in Q, is mapped upon a surface Σ of order 16 and class 4, the Cayley symmetroid quartic envelope with ten tropes. The two cubic curves are mapped upon two paired rational space sextics $\bar{R}_1(\tau)$, $\bar{R}_2(t)$ which are conjugate to the paired rational plane sextics $\bar{S}_1(\tau)$, $\bar{S}_2(t)$, respectively, i.e., plane sections of the space sextic are apolar to line sections of the *Conjugate* plane sextic. The symmetroid Σ is the locus of planes which cut the sextic $\bar{R}_i$ in catalectic sections. Similarly the jacobian $\bar{J}$ *counter* to J is mapped by the web $\bar{Q}$ upon a point symmetroid $\bar{\Sigma}$ *counter* to Σ, and C_1, C_2 upon rational space sextics $R_1(\tau)$, $R_2(t)$, counter to $\bar{R}_1(\tau)$, $\bar{R}_2(t)$, respectively, and conjugate to $S_1(\tau)$, $S_2(t)$, respectively.

5. *References.*—Meyer[3] has discussed the relation of J to the sextic $\bar{S}_2(t)$ and mentions the occurrence of counter sextics. Conner[4] considers the mapping from J to Σ and its connection with the paired rational sextics. The above introduction of the tetrad of rational sextics as defined by the sextics F, $\bar{F}$ of genus 4 is novel. Schottky,[5] beginning with the abelian theta functions of genus 4, derives a set of ten points in space which are the nodes of a quartic surface and merely states a characteristic property of this surface by which it can be identified with Σ. The writer[6] has shown that Σ can be transformed by regular Cremona transformation into only a finite number of projectively distinct symmetroids. These classes permute under the group (mod. 2) of integer transformations of the periods of the functions of genus 4. The analogous result for the plane

rational sextic involves a subgroup of the group (mod. 2) of the periods of the functions of genus 5. This indicates a connection (which we seek) of the functions of genus 4 and those of genus 5. Proceeding the other way Wirtinger[7] obtains the plane sextic of genus 4 as the locus of vertices of diagonal triangles of a linear series g_1^4 upon a ternary quartic ($p = 3$). This transition will be discussed later.

6. *The Covariant Conic* $\overline{R}(\tau)$ *of the Rational Plane Sextic* $\overline{S}_2(t)$.—From the existence in the net $\overline{Q}_1$ of a quadratic system of cones we conclude that the rational sextic $\overline{S}_2(t)$ has a covariant conic $K(\tau)$ such that the ten nodes of $S_2(t)$ determine upon $K(\tau)$ the ten pairs of nodal parameters of the sextic $\overline{S}_1(t)$ paired with the given sextic $\overline{S}_2(t)$. This theorem furnishes the bond between ten nodes as a ternary figure and ten nodes as a binary figure on the rational curve. The equation of the sextic in Darboux coördinates referred to the norm conic $K(\tau)$ is precisely that given in 3.

7. *The Perspective Cubics of* $\overline{S}_2(t)$. *The form* $\binom{113}{211}$. We denote by the symbol $\binom{i_1,\, i_2,\,\cdots\cdots}{k_1,\, k_2,\,\cdots\cdots}$ an algebraic form of order i_1 in the variables of an S_{k_1}, of order i_2 in the variables of an S_{k_2}, etc. Unless explicitly restricted these sets of variables are digredient. Thus $F = (a\tau)^3\ (\alpha t)^3$ is a form $\binom{33}{11}$. By polarizing F into $(a\tau_1)\ (a\tau_2)\ (a\tau)\ (\alpha t)^3$ and replacing the pair of parameters τ_1, τ_2 by the point x which they determine in the plane of $K(\tau)$ we obtain the $\binom{113}{211}$ form

$$(\pi x)\ (d\tau)\ (\delta t)^3$$

a general form of the orders indicated with nine absolute constants. For given τ this form determines a rational cubic envelope *perspective*[8] to the sextic $\overline{S}_2(t)$, i.e., line t of the cubic is on point t of the sextic. The sextic is the locus of the meets of corresponding lines of any two of the ∞^1 perspective cubics, and it has the equation $(\pi\pi'\xi)\ (dd')\ (\delta t)^3\ (\delta' t)^3 = 0$. Conversely given the sextic the family of perspective cubics is determined. Each cubic τ has three cusps whose parameters are given by $(\pi\pi'\pi'')\ (d\tau)\ (d'\tau)\ (d''\tau)\ (\delta\delta')^3\ (\delta'' t)^3 = 0$. This is $\overline{F} = (A\tau)^3\ (\mathrm{A}t)^3$ whence the cusp locus, $\overline{G}C(\tau)$, is birationally general and of genus 4. The equation of the cusp locus is the determinant of the coefficients of $(\pi x)\ (\pi' x)\ (d\tau)\ (d'\tau)\ (\delta\delta')^2\ (\delta t)\ (\delta' t)$ regarded as a form bi-quadratic in τ, t. Thus $\overline{G}C(\tau)$ is a sextic whose six nodes are the points for which the first minors of the above determinant vanish and these first minors furnish the nine linearly independent quartic adjoints of $\overline{G}C(\tau)$.

The curve of genus 4 has two special series g_1^3, residual with respect to each other in the canonical series. These appear in the normal form as the triads on the two sets of generators of the quadric containing the sextic. One of these series on $\overline{G}C(\tau)$ is the triads of cusps of perspective cubics of $\overline{S}_2(t)$. The web of adjoint cubics of $\overline{G}C(\tau)$ is furnished by the form

$$(\pi x)\ (\pi' x)\ (\pi'' x)\ (dd')\ (\delta\delta')\ (\delta'\delta'')\ (\delta\delta'')^2\ (\delta' t)\ (d''\tau) = 0,$$

t and τ being variable with the cubic of the web. For fixed τ and variable

t we have the pencil of adjoint cubics on the cusp triad of the perspective cubic τ.

The form $(\pi x)\ (d\tau)\ (\delta t)^3$ for fixed x and variable τ is a pencil of binary cubic. This pencil has two linear combinants[9]: $\bar{a} = (\pi x)\ (\pi' x)\ (dd^1)\ (\delta\delta')\ (\delta t)^2\ (\delta' t)^2$ and $\bar{b} = (\pi x)\ (\pi' x)\ (dd')\ (\delta\delta')^3$. The invariants i, j of the binary quartic $\bar{a}$ also are combinants. The invariant j is a sextic curve $\bar{d}$, the invariant i is $\bar{b}^2$. Hence the discriminant of $\bar{a}$ factors and the two factors $\bar{b}^3 + \bar{d}$ and $\bar{b}^3 - \bar{d}$ furnish the equations of the cusp locus $\overline{GC}(\tau)$ and the rational sextic $\bar{S}_2(t)$. We conclude further that there are 12 perspective cubics of $S_2(t)$ with flex points at the meets of $\bar{b}$ and $\bar{d}$. The sextics osculate at these points with the flex tangents as common tangents. The 12 flex points are the branch points on $\overline{GC}(\tau)$ of the function $t(\tau)$ defined by $\bar{F} = 0$. Thus a projective (but not a birational) peculiarity of $\overline{GC}(\tau)$ is that the 12 branch points of one of its series g^3_1 lie on a conic $\bar{b}$.

The parametric line equation of the conic $K(\tau)$ on which the nodes of $\bar{S}_2(t)$ determine the nodal parameters of $\bar{S}_1(\tau)$ is of degree four in the coefficients of $\binom{113}{211}$. Its symbolic form is $(\pi\pi'\pi'')\ (\pi''' x)\ (d'\tau)\ (\delta\delta')^3\ (\delta''\delta''')^3 \ \{ (d\tau)\ (d''d''') + 2(d''\tau)\ (dd''') \} = 0$.

With reference to the cubic space curves $C_1(\tau)$, $C_2(t)$ the point x determines an axis l_x of $C_1(\tau)$ on planes τ_1, τ_2; τ is the third plane of $C_1(\tau)$ on a point y of l_x; and t a plane of $C_2(t)$ on y. Then to points x on $\overline{GC}(\tau)$ there correspond axes of C_1 on points of C_2 and to the nodes of $\overline{GC}(\tau)$ the six axes of C_1 which are chords of C_2; to points x on $\bar{S}_2(t)$ there correspond axes of C_1 on planes of C_2, and to the nodes of $\bar{S}_2(t)$ the ten common axes of C_1, C_2. If x_0 is a node of $\bar{S}_2(t)$ the form $(\pi x_0)\ (d\tau)\ (\delta t)^3$ factors into $(l_0\tau)\ (\lambda_0 t)\ .\ (q_0 t)^2$ where $(q_0 t)^2$ is the pair of nodal parameters. The ten forms $(l_0\tau)\ (\lambda_0 t)$ will appear later in connection with the symmetroid. Other covariants of the $\binom{113}{211}$ form are easily interpretable with reference to C_1, C_2. Thus $\bar{a}$ furnishes the four parameters t of tangents of C_2 which meet the axis l_x of C_1, and b determines the axes l_x of C_1 which are in the null system of C_2.

From the definition of perspective curves the line t' of the perspective cubic $(\pi x)\ (d\tau)\ (\delta t')^3$ will cut the sextic $(\pi\pi'\xi)\ (\delta t)^3\ (\delta' t)^3\ (dd')$ in the point $t = t'$. On forming the incidence condition of line and point, removing the factor (tt'), and setting $t' = t$, we obtain

$$(\pi\pi'\pi'')\ (\delta\delta')\ (\delta t)^2\ (\delta' t)^2\ (\delta'' t)^3\ (d'd'')\ (d\tau)$$

which furnishes the seven contacts[10] t of the perspective cubic with the sextic. This is a form $\binom{71}{11}$ of general type containing nine absolute constants which will appear later.

[1] This investigation has been pursued under the auspices of the Carnegie Institution of Washington, D. C.

[2] H. S. White, these PROCEEDINGS, 2, 1916 (337).

[3] Meyer, *Apolarität und Rationale Curven*, pp. 320–47.

[4] Conner, *Amer. J. Math.*, **37**, 1915 (29).
[5] Schottky, *Acta Math.*, **27**, 1903 (235).
[6] Coble, *Amer. J. Math.*, **41**, 1919 (243).
[7] Wirtinger, *Math. Ann.*, **27**, 1892 (261); *Untersuchung über Thetafunctioners*, Leipzig (1895).
[8] Coble, *Amer. J. Math.*, **32**, 1910 (333).
[9] Shenton, *Ibid.*, **372** 1915 (247).
[10] Cf. Coble, *l. c.*, p. 352.

MELANOVANADITE, A NEW MINERAL FROM MINA RAGRA, PASCO, PERU

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Late in 1920 Mr. W. Spencer Hutchinson, Consulting Engineer for the Vanadium Company of America, brought to my attention three specimens of a mineral collected by him at Mina Ragra, Peru. He suspected that it was a new mineral, and this opinion was proved correct by chemical and optical examination. The formula is $2CaO.\,3V_2O_5.2V_2O_4$ and I wish to propose for it the name of Melanovanadite, in allusion to it being practically the only vanadium mineral of a deep black color.[1]

The mineral occurs in acicular bunches on black brecciated shale, the individual crystals being at most 3 mm. long.

The greater thickness of the needles is about 0.5 mm. ranging down to 0.1 and 0.01 mm. The color is black, luster almost submetallic, streak very dark reddish brown. The hardness is 2.5 the specific gravity 3.477 at 15° C. The habit of the crystals is prismatic, parallel to *c*, with monoclinic symmetry. The principal faces consist of a flat, striated prism, the longer diagonal being parallel to the *b* axis, minor pinacoidal faces, and usually well developed terminal faces of pyramids and smaller domes. The crystals have a perfect cleavage parallel to (010).

Under the microscope the crystals remain black except in very thin prisms which are translucent with brown color.

Flat cleavage pieces parallel to the clinopinacoid only become translucent when the thickness is about 0.003 mm. and then show maximum extinction of about 15 °. Resting on the prism (100) the crystals become brown translucent with a thickness of about 0.03 mm. and then show lower extinctions of 12° to 13°, while these resting more nearly on the orthopinacoid extinguish at lower angles. The perfect cleavage being perpen-

[1] The ending "vanadite" is an obsolete form of "vanadinite," but there can scarcely be any objection to using this form in the present case.